Roll No. ....

DD-462

M. A./M. Sc. (Second Semester) EXAMINATION, May/June, 2020

MATHEMATICS

Paper Fourth

(Advanced Complex Analysis—II)

Time : Three Hours

Maximum Marks : 80

Note : Attempt *two* parts from each Unit. All questions carry equal marks.

Unit—I

1. (a) If $|z| \leq 1$ and $p \geq 0$, then show that :

$$\left|1 - E_p(z)\right| \leq |z|^{p+1}$$

(b) Let $S = \{z \in C : a \leq \operatorname{Re} z \leq A\}$, where $0 < a < A < \infty$. Then show that for every $\in > 0$ there is a number K such that for all z in S :

$$\left|\int_{\alpha}^{\beta} e^{-t}\, t^{z-1}\, dt\right| < \in$$

whenever $\beta > \alpha > k$.

(c) Let r be a rectifiable curve and let K be a compact set such that $K \cap \{r\} = \phi$. Let f be a continuous function on $\{r\}$ and let $\in > 0$ be given. Then show that there is a rational function R(z) having all its poles on $\{r\}$ and such that :

$$\left| \int_r \frac{f(w)}{w - z} dw - \mathrm{R}(z) \right| < \in$$

for all z in K.

Unit—II

2. (a) Let $r : [0,1] \to \mathrm{C}$ be a path and let $\{(f_t, \mathrm{D}_t) : 0 \leq t \leq 1\}$ be an analytic continuation along r. For $0 \leq t \leq 1$ let R(t) be the radius for convergence of the power series expansion of f_t about $z = r(t)$. Then show that either $\mathrm{R}(t) \equiv \infty$ or **R** : [0, 1] $\to$ (0, ∞) is continuous.

(b) Show that the function $f_1(z) = 1 + 2 + 2^2 + 2^3 + ...$ can be obtained outside the circle of convergence of the power series.

(c) Define Analytic Continuation. If the radius of convergence of the power series :

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is non-zero finite, then show that f (z) has at least one singularity on the circle of convergence.

Unit—III

3. (a) Define Poisson kernel. Show that the Poisson kernel $\mathrm{P}_r(\theta)$ satisfies the following properties :

(i) $\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{P}_r(\theta) \, d\theta = 1$

(ii) $\mathrm{P}_r(\theta) > 0$ for all θ, $\mathrm{P}_r(-\theta) = \mathrm{P}_r(\theta)$ and P_r is periodic in θ with period 2π.

(b) Let G be a region and $f : \partial_\infty G \to \mathbf{R}$ a continuous function. Then show that $u(z)$ = Sup $\{\phi(z) : \phi \in P(f, G)\}$ defines a harmonic function u in G.

(c) To state and prove Harnack's theorem for harmonic functions.

Unit—IV

4. (a) If $f(z)$ is analytic within and on the circle r such that $|z| = R$ and if $f(z)$ has zeros at the points $a_i \neq 0$, $(i = 1, 2, \ldots, m)$ and poles at $b_j \neq 0$, $(j = 1, 2, \ldots, n)$ inside r, multiple zeros and poles-being repeated, then show that :

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(\mathrm{Re}^{i\theta}\right| d\theta = \log|f(0)| + \sum_{i=1}^{m} \log \frac{R}{|a_i|} - \sum_{j=1}^{n} \log \frac{R}{|b_j|}$$

(b) Define order of an Entire Function. Find the order of polynomial $P(z) = a_0 + a_1 z + \ldots + a_n z^n, a_n \neq 0$.

(c) Use Hadamard's factorization theorem to show that :

$$\sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

Unit—V

5. (a) Define Bloch's constant. Let f be an analytic function in a region containing the closure of the disc $D = \{z : |z| < 1\}$ and $f(0) = 0$, $f'(0) = 1$. Then show that $f(D)$ contains a disc of radius L.

(b) For each α and β, $0 < \alpha < \infty$ and $0 \le \beta \le 1$, there is a constant $C(\alpha, \beta)$ such that if f is an analytic function defined in some simply connected region containing $\overline{B}(0,1)$ that omits the values 0 and 1 and such that $|f(0)| \le \alpha$; then show that :

$$|f(z)| < C(\alpha, \beta) \text{ for } |z| \le \beta.$$

(c) Let $f \in y$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then show that :

(i) $|a_2| \le 2$

(ii) $f(U) \supset D\left(0; \frac{1}{4}\right)$